Homework I

Due Date: 28/02/2022

Exercise 1 (2 points).(i) Find the the solutions that depend only on r of the equation $\partial_x^2 u + \partial_y^2 u + \partial_z^2 u = u$.
(ii) Solve $\partial_x^2 u + \partial_y^2 u + \partial_z^2 u = 0$ in the spherical shell $0 < a < r < b$ with the boundary conditions $u = A$ on $r = a$ and $u = B$ on $r = b$, where A and B are constants.
(iii) Solve $\partial_x^2 u + \partial_y^2 u = 1$ in the annulus $a < r < b$ with $u(x, y)$ vanishing on both parts of the boundary $r = a$ and $r = b$.
(iv) Solve $\partial_x^2 u + \partial_y^2 u + \partial_z^2 u = 1$ in the spherical shell $a < r < b$ with $u(x, y, z)$ vanishing on both the inner and outer boundaries.
(v) Show that there is no solution of

$$\Delta u = f \text{ in } D \quad \text{and} \quad \frac{\partial u}{\partial n} = g \text{ on } \partial D.$$

in three dimensions, unless

$$\int_D f(x, y, z) \mathrm{d}x \mathrm{d}y \mathrm{d}z = \int_{\partial D} g(x, y, z) \mathrm{d}S.$$

Exercise 2 (1 point). Consider the following problem

$$\begin{cases} \Delta u = 0, & \text{in } D, \\ u = h, & \text{on } \partial D, \end{cases}$$

where $D = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1\}$.

(i) Show that if $h \geq 0$, then $u > 0$ in D unless $h = 0$.

(ii) Let $u(0) = 1$ and $h \geq 0$. Show that

$$\frac{1}{3} \leq u(x, y) \leq 3$$

for all $x^2 + y^2 = \frac{1}{4}$.

Exercise 3 (3 points). (i) Consider the following problem

$$\begin{cases} \Delta u = u^3, & \text{in } D, \\ \dfrac{\partial u}{\partial n} + a(x) u = h, & \text{on } \partial D, \end{cases} \tag{1}$$

where $a(x) \geq 0$. Show that the solution to (1) (if exixts) is unique.
(ii) Consider the following problem

$$\begin{cases} \Delta u - b(x) u = f(x), & \text{in } D, \\ u = h, & \text{on } \partial D, \end{cases} \tag{2}$$

where b is positive. Let u be a C^2 function.
(a) Define an energy functional $E[u]$ associated with (2).
(b) Show that u is a solution to (2) if and only if

$$E[w] \geq E[u], \quad \text{for all } w \in C^2 \text{ with } w = h \text{ on } \partial D.$$

(iii) Given $h(x)$, $g(y)$, find a solution formula for the following problem

$$\begin{cases} \Delta u = 0, & \text{in } Q = \left\{(x, y) \in \mathbb{R}^2 : x > 0, y > 0\right\}, \\ u(x, 0) = h(x), & \text{for } x > 0, \\ u(0, y) = g(y), & \text{for } y > 0. \end{cases}$$

Hint: Find the Green's function in Q first.

Exercise 4 (3 points). Invariance of the 3D Laplace' equation.
(i) Show that if $y \in \mathbb{R}^3$ and u is harmonic, then $v(x) := u(x-y)$ is also harmonic.
(ii) Show that if $\lambda > 0$ and u is harmonic, then $v(x) := u(\lambda x)$ is also harmonic.
(iii) Show that if O is an orthogonal 3×3 matrix and u is harmonic, then $v(x) := u(Ox)$ is also harmonic.
(iv) Show that if u is harmonic, then $v(x) := \frac{1}{|x|} u(\frac{x}{|x|^2})$ is also harmonic. This is so called Kelvin transform.

Exercise 5 (1 point) Let $B_1 \in \mathbb{R}^3$ be the unit ball. Show that any smooth vector field $\vec{E} = (E_1, E_2, E_3)$ over B_1 can be written as $\vec{E} = \vec{F} + \vec{G}$ such that $\mathrm{curl}\vec{F} = 0$ and $\mathrm{div}\vec{G} = 0$.